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Analysis of Heat Exchangers for Dilution Refrigerators

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Numerical calculations of the behavior of dilution refrigerator heat exchangers are discussed and some results for both discrete and continuous exchangers are presented. It is shown that thermal conductance along the stream is negligible for a typical continuous exchanger of the coaxial tube type, but becomes a dominant feature of a typical discrete exchanger operating below about 50 mK and degrades the performance considerably. A simple design change can be made that reduces the conductance along the liquid and improves the performance of such an exchanger. A simple means of determining whether conductivity is important in either continuous or discrete exchangers is given.

INTRODUCTION

CONSTRUCTION details for counterflow heat exchangers used in continuously operating dilution refrigerators have been discussed by several authors. The main design considerations are that the liquid volume be as small as possible so equilibrium can be established rapidly when the temperature is changed, the impedance to the flowing liquids be small, the thermal conductivity of the exchanger between the two streams be adequate, and the heat transfer areas be as large as possible to overcome the effects of the Kapitza resistivity. The lowest temperature achievable in the mixing chamber depends upon the effectiveness of these heat exchangers.

Some analyses of these exchangers have been done in the past, but various simplifying approximations, such as zero or infinite liquid thermal conductivities, were made so that the differential equations governing heat transfer could be solved analytically. However, to analyze some types of exchangers and, in any case, to include proper conductivities and heat capacities, the differential equations must be solved numerically. To predict or optimize exchange performance, proper conductivities, heat capacities, etc., must be used.

Two types of counterflow exchangers are commonly used in dilution refrigerators. The discrete type, first used by Wheatley et al., consists of blocks containing two cavities (see Fig. 1) into which a high surface area material is sintered. High conductivity copper is generally used for both the filler and the body. The exchanger usually consists of four to six of these blocks connected together with short lengths of low conductivity tubing such that the incoming stream passes through one cavity and the exit stream passes through the other cavity of each block. The other type is the continuous exchanger, in which there are no discontinuous changes in physical parameters over the full

![Fig. 1. Schematic of a copper block exchanger showing lengths L and T, flow directions n, and temperatures. The temperatures T_{cl}, T_{do}, T_{di}, and T_{o} are those of the concentrated inlet and outlet and dilute inlet and outlet streams, respectively.](image)

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length of the exchanger. An example of this type is the coaxial tube exchanger first used in dilution refrigerators by Ehnholtz et al., which consists of a long, small-diameter Cu-Ni tube loosely fitted inside a larger Cu-Ni tube. The incoming stream passes through the central tube while the return stream passes through the annulus. This particular type is gaining popularity because it is simple to construct and has a small liquid volume. In this paper, more detailed analyses of the discrete and continuous exchangers are presented.

I. GENERAL CONSIDERATIONS

The two coupled differential equations describing both types of heat exchangers can be derived from the steady state flow equation, for the fluid on each side, is

\[ A_j \left[ \frac{d^2 T_{i, j}}{d x^2} + \frac{d T_{i, j}}{d x} \right] \]

\[ \text{heat conduction} \]

\[ \int_{T_i}^{T_j} \frac{d T}{\rho_j \text{viscous heating}} - \frac{d T_j}{d x} \text{enthalpy change} = 0, \]

(1)

where \( j \) is a general label that will be replaced with \( c \) when the concentrated stream is considered, or by \( d \) when the dilute stream is considered. The Joule-Thomson cooling of \(^3\)He in the dilute stream due to an osmotic pressure drop across the exchanger is negligible compared with the viscous heating that would occur simultaneously. Using the appropriate boundary conditions, this set of equations is solved numerically with a digital computer to determine the temperature profiles of the concentrated and dilute streams, \( T_c(x) \) and \( T_d(x) \), respectively. A library subroutine employing the Runge-Kutta method is used in the program. The various parameters are \( x \), the position in the exchanger along the direction of flow; \( A_j \), the liquid cross sectional area perpendicular to the direction of flow; \( \kappa_j \), the liquid thermal conductivity; \( \sigma_{j, d} / d x \), the heat transfer surface area per unit length of exchanger; \( \rho_j \), Kapitza resistivity; \( T_b \), the temperature of the heat exchanger body; \( n_a \), \(^3\)He molar flow rate; and, \( C_j \), the molar specific heat of the liquid stream. The viscous heating of the moving liquids, \( W_j \), may cause some heating at the lowest temperatures but will be considered negligible here.

Since mixer temperatures of 0.01 K or higher are being considered here, thermal relaxation effects between \(^3\)He quasiparticles and phonons in the dilute solution are considered small and not included. Analytic expressions are used for \( \rho_j, \kappa_j, \) and \( C_j \) in the solution of Eq. (1).

The expression for Kapitza resistivity,

\[ \rho_j = \left[ \frac{(20 \times 10^{-4})/T_j^4}{cm^2 \cdot K/\mu W} \right], \]

(2a)

was determined from the data of Anderson et al., and is valid between 0.01 and 0.13 K. In most cases discussed here, it is assumed that this dependence continues to 0.7 K. In some cases, which will be discussed further below, the equation,

\[ \rho_j = \left( \frac{2.4}{T_j^4} + \frac{1.55}{T_j^{6.5}} \times 10^{-5} \left( \frac{cm^2 \cdot K}{\mu W} \right) \right), \]

(2b)

is used from 0.13 < \( T_j < 0.7 \) K. This expression more correctly fits the data of Anderson et al., in this temperature range.

The boundary resistance between a nearly saturated dilute solution and copper has been measured by Wheatley et al.3 up to 0.1 K. There is some deviation from a \( T^{-4} \) dependence, but for this work it is assumed that

\[ \rho_d = \left[ \frac{(7 \times 10^{-3}/T_d)}{cm^2 \cdot K/\mu W} \right], \]

(3)

up to 0.2 K. Since there are presently no data on the commonly used 70/30 Cu-Ni alloys, Eqs. (2a), (2b), and (3) are used for the boundary impedance of this material also.

For the liquid thermal conductivity, \( \kappa_j \), the results of Abel et al.,5,6 were fitted by

\[ \kappa_j = \left( \frac{3.48}{T_j} + 31.4 + 58.1 T_j \right) \left( \frac{cm \cdot K}{\mu W} \right), \]

(4)

An expression for the dilute side heat capacity was taken from the data of Abel et al.,5,6 The expression for the concentrated \(^3\)He, \( C_n \), was taken from calculations by Radebaugh.11 The term \( C_n \) is the heat capacity at constant osmotic pressure which is weakly dependent on the mixer temperature \( T_m \). For all calculations presented here, \( T_m \) is small enough that the error is small if \( T_m \) is assumed to be zero. The dilute stream flow impedance must be small also since \( C_n \) is an equilibrium value under zero flow conditions, i.e., no pressure drop in the dilute stream. Preliminary measurements of \( C_n \) indicate agreement to better than 10% between the calculated and experimental results for \( T_d \) less than 0.2 K. Analytic expressions for \( C_d \) have been determined by polynomial fitting of the calculated values of \( C_n \) for \( T_d = 0 \). The analytic expressions used for \( C_d \) and tables of the liquid enthalpies are given in the Appendix.
II. DISCRETE EXCHANGERS

A. Method of Calculation

The calculation is simplified considerably if it is assumed that the copper body is at constant temperature. This assumption is generally made. An unpublished report on some experimental work by Ekmahl and Wheatley indicates that this assumption is reasonable for the heat fluxes encountered in these heat exchangers. Equation (1) is solved for each side of the exchanger (see Fig. 1) subject to the condition that the net heat transferred to the heat exchanger body is zero.

The optimum volume ratio of the two sides of a discrete exchanger is obtained by specifying some volume, \( V_s \), of pure \(^4\)He liquid that is to be divided between the two sides of the exchanger in such a way that the heat transferred is a maximum. The liquid cross sectional areas of the heat exchanger are expressed in terms of \( V_s \) and \( f \), the fraction of the \(^4\)He contained in the dilute side. These areas are

\[
A_e = (1 - f)V_s/L
\]

(6a)

for the concentrated side, assuming only pure \(^4\)He present, and

\[
A_d = (fV_s/L)[27.58/X_e + 7.6 + 1.65X_e^2]/36.83
\]

(6b)

for the dilute side. The exchanger length is \( L \) and the expression in brackets is the volume of dilute solution in cubic centimeters per mole of \(^4\)He, where \( X_e \) is the \(^4\)He concentration, and 36.83 cm\(^3\) is the molar volume of pure \(^4\)He.\(^{(13)}\) The optimum value of \( f \) is found by maximizing the heat transferred in the exchanger. The ratio of dilute side to concentrated side volume can be calculated from \( f \) and the molar volumes.

The condition on the thermal gradient at the exchanger tube interface is

\[
A_i(dT_j/dx)_{\text{tube}} = A_j(dT_j/dx)_{\text{exchanger}}
\]

(7)

where \( A \) is the tube cross section. The interconnecting tubes between exchangers and the connecting tubes to the still and mixer are sufficiently small so that the heat conducted through them can be neglected. Thus, at the exit end of the heat exchanger cavity, \( dT_j/dx = 0 \) by Eq. (7) for both concentrated and dilute streams because the \( dT_j/dx \) in the tube at this end must be zero when there is no conductance between exchangers. Within the inlet tubes, the temperature of the concentrated or of the dilute stream can be higher or lower, respectively, than the liquid in the exchanger. Conductance of the liquid in the tube changes the temperature of the stream just before it enters the exchanger. The slopes at the inlet end are not known and starting the problem at this end requires an iteration to arrive at the exit end with \( dT_j/dx = 0 \). This iteration is avoided by starting the calculation at the exit end.

The dilute side behavior is calculated first; hence \( j = d \) in Eq. (1). Values for \( T_b \) and \( T_{d_o} \), where \( T_{d_o} \) is the dilute side outlet temperature (see Fig. 1), are assumed and the calculated value is carried from \( x = 0 \) to \( L \). Since it is desired that the mixer temperature \( T_m \) be fixed, the calculation for the lowest heat exchanger is iterated with respect to \( T_b \) until the change in enthalpy of the incoming stream prior to entering the exchanger is equal to the heat transferred out of the exchanger via liquid conduction to the incoming stream. That is, the equation,

\[
H_d(T_{d_e}) - H_d(T_e) = \kappa_d \left[ \frac{A_d}{\eta_s} \frac{dT_d}{dx} \right]_{x=L}
\]

(8)

is satisfied. The term \( H_d(T) \) is the dilute stream enthalpy, \( T_{d_e} \) is the temperature of the liquid in the input end of the heat exchanger, and \( T_{d_o} = T_m \). For each choice of \( T_{d_o} \), a unique value for \( T_{d_e} \) is calculated.

If it is assumed that the concentrated stream is 100% \(^4\)He and there is no heat input to the mixer, then, from an enthalpy balance on the mixer,\(^{(15)}\)

\[
T_{d_o} = T_{d_1}/0.36
\]

(9)

where \( T_{d_o} \) is the concentrated outlet temperature. The concentrated side behavior is calculated using Eq. (1) with \( j = c \) and Eq. (9). The final value of \( T_b \) from the dilute side calculation is the \( T_b \) used for the concentrated side calculation. The enthalpy changes on the two sides generally will not be equal, so a second iteration is done varying \( T_{d_o} \) until these changes are equal. For each change in \( T_{d_o} \) the dilute side must be recalculated as described above to find the new value of \( T_b \). The value of \( T_{d_o} \) is chosen such that the enthalpy changes of the concentrated and dilute streams are equal.

The next exchanger up the line can now be calculated using the \( T_{d_o} \) calculated above as the new \( T_{d_1} \), using the
rate is \(3 \times 10^{-4}\) moles/sec. The first example has a flow rate and surface area closer to typical values, although the surface area is given a value smaller than actual areas as there is some evidence\(^7\) to suggest the effective areas may be smaller than the actual areas.

By solving the problem for various values of the fraction \(f\) in Eq. (6), and plotting \(f\) vs the heat transferred between streams, the optimum ratio of volumes on the two sides of the exchanger can be found. Figure 2 is a plot of the heat transferred as a function of \(\sigma_d/\sigma_s\), the ratio of surface areas for the lowest temperature with \(\sigma_s = 400\) cm\(^{-1}\) and flow rate of \(2 \times 10^{-5}\) moles/sec. The ratio \(\sigma_d/\sigma_s\) is equivalent to the ratio of volumes when the same sponge particle size and density are used on each side. The optimum ratio occurs at \(\sigma_d/\sigma_s = 2.1\) \((f = 0.15)\) for the lowest temperature heat exchanger, but the sensitivity of the transferred heat to this ratio is low. For \(T_{cs} \leq 0.1\) K the maximum occurs at the same value of \(f\), but \(\sigma_d/\sigma_s\) is now 2.5 due to the decrease of \(^4\)He concentration in the dilute solution. The heat transfer at a \(T_{cs}\) of 0.1 K is less sensitive to \(\sigma_d/\sigma_s\) than is shown by the curve of Fig. 2, for which \(T_{cs}\) is about 0.028 K.

In Fig. 3, the behavior of the lowest temperature exchanger with \(T_{cs} = 10\) mK is shown, where \(\sigma_s = 400\) cm\(^{-1}\) and \(\sigma_d/\sigma_s = 2.1\) \((f = 0.15)\). Curves for \(T_{c}, T_{d1}\), and \(T_{d2}\) are shown as a function of \(x\) through the body, with an extrapolation of \(T_{d1}\) and \(T_{d2}\) into the outside tubes shown also. The incoming stream temperature profiles immediately outside the exchanger are estimated using the slopes determined from Eq. (7), along with \(T_{d1}\) and \(T_{d2}\), which are equivalent to the outlet temperatures of the adjacent exchangers. The conductivities of the liquids are so high that \(\Delta T_d\) is less than 0.4 mK and \(\Delta T_s\) less than 0.7 mK inside the exchanger. Most of the temperature change occurs in the interconnecting tube just prior to entering the heat exchanger for this case. The \(\Delta T_d\) of the liquids in the exchanger are small compared to \(T_{d1}-T_{d2}\). This is true when the conductance through the liquid is very large compared to the conductance to the walls, i.e.,

\[
A_p \frac{\rho_p}{L \sigma} \gg 1.
\] (10)

The heat exchanger performance is relatively independent of \(K_j\) when Eq. (10) is true. For the exchanger with the smaller surface for the concept isothermal and instantaneous heat exchanger was used.

### Table I. Calculated Inlet and Outlet Temperatures and Effectiveness \(R\) of the Model Exchanger

The temperatures \(T_{c1}, T_{c0}, T_{d1},\) and \(T_{d0}\) are those of the concentrated stream inlet and outlet and the dilute stream inlet and outlet, respectively.

<table>
<thead>
<tr>
<th>Exchanger</th>
<th>Total vol of (^4)He</th>
<th>(T_{c1})</th>
<th>(T_{c0})</th>
<th>(T_{d1})</th>
<th>(T_{d0})</th>
<th>(R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. 1</td>
<td>1 cm(^3)</td>
<td>40.5 mK</td>
<td>27.78 mK</td>
<td>10.0 mK</td>
<td>16.77 mK</td>
<td>36%</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>72.86</td>
<td>40.3</td>
<td>16.77</td>
<td>31.25</td>
<td>59</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>185.2</td>
<td>72.86</td>
<td>31.25</td>
<td>70.03</td>
<td>95</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>672.7</td>
<td>183.2</td>
<td>70.03</td>
<td>180.25</td>
<td>&gt; 99</td>
</tr>
<tr>
<td>1 (Composed of two)</td>
<td>2</td>
<td>48.8</td>
<td>27.78</td>
<td>10.0</td>
<td>20.73</td>
<td>58</td>
</tr>
<tr>
<td>1 (3 partitions)</td>
<td>1</td>
<td>51.66</td>
<td>27.78</td>
<td>10.0</td>
<td>22.06</td>
<td>65</td>
</tr>
</tbody>
</table>

\(\sigma_s = 3 \times 10^{-4}\) moles/cm\(^2\).

---

**Fig. 4.** Calculated heat exchanger performance and effectiveness \(R\) for the model exchanger. The data for the concept isothermal and instantaneous heat exchanger was used.
smaller surface area, the left side of Eq. (10) is about 3.4 for the concentrated side. The liquids may be considered isothermal and, to avoid solving Eq. (1), a set of simultaneous algebraic equations can be used to roughly determine the exchanger behavior. In Fig. 4, the results are shown for the larger surface area exchanger. The left side of Eq. (10) is now approximately 0.2 for the concentrated side and $\Delta T$ is quite large. In this case the heat exchanger behavior can only be determined from solutions of Eq. (1).

The dotted curves in Fig. 3 show the behavior at the $\kappa=0$ limit for $T_m = 10$ mK. Similar curves exist, but are not shown, for the $\kappa=6000$ cm$^{-1}$ exchanger, and give a $T_c$ of 0.0085 K. The heat transferred between the streams in the $\kappa=0$ limit is about three times as great as in the $\kappa \neq 0$ case. From this result it is apparent that the large liquid conductivity can considerably degrade the exchanger performance.

Table I shows the inlet and outlet temperatures of some heat exchangers. The relative effectiveness, listed in the last column, compares the performance of one of these exchangers with a discrete exchanger capable of producing the maximum possible change of $T_c$, but with the same $T_c$. This relative effectiveness, $R$, is defined by

$$ R = \frac{(T_{ci} - T_{co})}{[(T_{ci})_{perfect} - T_{co}]} $$

The temperature $T_{co}$ rather than $T_{ci}$ is held fixed in this definition of $R$ since the calculations must be done with a set $T_{co}$. The perfect discrete heat exchanger has an infinite surface area; thus, $T_{co} = T_{di} = T_{in}$. The first four lines of Table I show the results of calculations of successive heat exchangers from a mixer temperature of 10 mK up to a still temperature of 0.67 K for the dimensions and flow rate described above. Five exchangers of the type described above with $\sigma_c = 400$ cm$^4$ are required for a 0.7 K still temperature. A 1 cm$^3$ volume of $^3$He for the highest temperature exchanger is so excessively large that $T_b$, $T_{do}$, and $T_{co}$ are nearly equal and convergence difficulties were encountered in the iterative parts of the solution. The volume of $^3$He was reduced by a factor of 4 for this exchanger to aid convergence. In practice, five exchangers have been required.

The relative effectiveness is high for the warmest exchanger shown in Table I, but decreases quite rapidly with decreasing $T$. The fifth line shows the change in $R$ upon doubling the size ($V_2 = 2$ cm$^3$) of the coldest exchanger, whereas the sixth line shows $R$ for a system of two exchangers of $V_2 = 1$ cm$^3$ each. The $R$ given for this combination is relative to a single discrete exchanger. As would be expected, it is considerably more effective to add volume as additional exchangers rather than enlarging the existing exchangers.

It is apparent from Fig. 3 that the performance of a heat exchanger can be improved by reducing conductance through the liquid. Making the exchanger long and small in cross section can accomplish this, but the flow impedance would probably be too high. The effect of the liquid conductivity in the discrete heat exchanger can be reduced rather easily by inserting several tight fitting disks or partitions into the chamber, dividing the sintered material into sections as shown in Fig. 5. These disks would be about 1 mm thick and have one hole through them whose inside diameter is the same as or slightly smaller than that of the interconnecting tubing. These hole dimensions are such that the liquid thermal conductance between compartments is small and will be neglected for purposes of this calculation. Figure 6 shows the calculated temperatures as a function of $x$ in the exchanger in which $\sigma_c = 400$ cm$^4$, but now with three disks inserted. This is compared to the

![Fig. 4. Calculated liquid and body temperature profiles for the same discrete exchanger shown in Fig. 3, but with $\sigma_c = 6000$ cm$^4$ and $\dot{n}_c = 3 \times 10^{-4}$ moles/sec. $E_c = 0$.](image-url)
III. CONTINUOUS EXCHANGER

A. Method of Calculation

Continuous exchangers will be divided into two categories: ideal and nonideal. The former has sufficiently small liquid and tube cross sections, such that the heat conducted parallel to the flow is negligible compared to the total heat transferred; i.e.,

\[ \sum_{j} \frac{A_{j} \kappa_{j}}{L \ln_{a} C_{j}} \ll 1, \]

where \( L \) is the total length of the continuous exchanger. The index \( j \) is summed over \( c, d, \) and \( b \) which refer to the concentrated stream, dilute stream, and body, respectively. However, the heat capacity \( C_{b} \) should be best be given a value equal to \( C_{d} \). The inequality above can be written as

\[ \sum_{j} \frac{B_{j} \kappa_{j}}{L \ln_{a}} \ll 1, \]

(12)

where \( B_{j} = \kappa_{j}/C_{j} \). The term \( B_{j} \) is a weak function of temperature and can be approximated by

\[ B_{j} = 4 \times 10^{-5} \text{ moles} \cdot \text{cm}^{-1} \cdot \text{sec}^{-1} \]

for both the dilute and concentrated liquids. For channel sizes the order of 50 \( \mu \) \( \text{m} \), \( B_{d} \) is reduced by a factor of only about 2 due to boundary effects on \( \kappa_{a} \). For the ideal continuous exchanger, the conductivity terms in Eq. (1) are neglected.

A continuous exchanger for which Eq. (12) is not true is considered nonideal. The value of the term on the left side of Eq. (12) can serve as a crude approximation to the deviation from the behavior of the ideal continuous exchanger, provided the term is still somewhat smaller than 1. The exact behavior in the nonideal case can be found only by solving Eq. (1) with all terms included.

The behavior of an exchanger with no heat conduction along the tube walls is found when the differential Eqs. (1) for \( T_{d} \) and \( T_{c} \) are solved simultaneously. This solution is subject to the condition that the net heat to the intermediate wall is zero at all points along the exchanger. That is,

\[ \frac{d x}{d \tau_{c}} \int_{\tau_{c}}^{\tau_{d}} \frac{dT}{\rho_{c}} = \frac{d x}{d \tau_{d}} \int_{\tau_{d}}^{\tau_{c}} \frac{dT}{\rho_{a}}. \]

(15)

This can be integrated analytically when \( \rho_{j} \) is given by Eqs. (2a) and (3) and the result can be solved for \( T_{d} \). The integral can be done analytically when \( \rho_{j} \) is given by Eq. (2b), but the integrated equation must be solved for \( T_{d} \) by iteration.

The solution at the mixer end of the exchanger is subject to the condition

\[ H_{d}(T_{d}) \text{ solubility curve} = \frac{Q}{\pi a} + H_{c}(T_{c}), \]

(14)

where the first term is the solubility curve, and the second term is the exchange term in rate Eq. (1) with \( \theta_{d} = T_{d} \).
where the first term is the dilute solution enthalpy on the solubility curve, $Q$ is the heat into the mixer due to thermal leakage and experimental heat loads, and the last term is the enthalpy of the pure $^4$He entering the mixer. Equation (14) is satisfied by iterating the problem with respect to $T_{do}$.

### B. Results and Discussion

Particular emphasis is given in this work to the small diameter coaxial tube exchangers. Since the tube diameters used in coaxial heat exchangers are generally quite small, the exchanger can almost always be considered ideal. The left side of Eq. (12) is usually the order of $10^{-3}$ for both liquid streams and even less than $10^{-5}$ for both the tube walls. Since data are not available for the boundary resistance between the liquids and stainless steel or Cu-Ni tubing generally used in coaxial tube exchangers, it is assumed that the heat transfer coefficients have the boundary resistance of copper given by Eqs. (2a), (2b), and (3). In addition, it is assumed that no temperature gradients exist in a direction transverse to the flow in either stream or in the intermediate tube and that there is negligible conductivity in the tube walls parallel to the flow.

Calculations have been done for a heat exchanger 1 m long consisting of a 2 mm o.d. x 0.1 mm wall and a 1 mm o.d. x 0.1 mm wall tube, and a 1.5 m long exchanger of a 1.20 mm o.d. x 0.076 mm wall and a 0.4 mm o.d. x 0.076 mm wall tube. These sizes are typical of heat exchangers in current use. Conductivity effects in these exchangers are negligible at normal flow rates. However, it is of interest to compare the deviation predicted by Eq. (12) with the rigorous result found from a solution of Eq. (1) with and without conductivity terms. Only conductivity in the dilute liquid was considered, since including conductivity in both streams complicates the problem considerably and gives little additional information, as the concentrated stream conductance is usually much smaller. The value for $k_\sigma$ used in Eq. (1) is given by

$$k_\sigma = (4.43 T_T^{-0.06} + 4.87 \times 10^2 T_T^{0.81} - 2.2 \times 10^5) (\frac{\mu W}{cm \cdot K}),$$

where $T_T$ is in degrees Kelvin. This expression approximately fits the data of Abel et al.\cite{11,14} interpolated to the proper concentration. This conductance in the dilute stream for the 1 m long exchanger raises the mixer temperature by $80 \mu K$ at $T_m = 16 \ mK$ when the flow rate is $5 \times 10^{-6}$ moles/sec. This is a deviation of $0.5\%$, and the value of $0.2\%$ from the left side of Eq. (12) is in rough agreement. Increasing the cross sectional area of the dilute stream by a factor of 100 increases the actual deviation to 14\% compared with $20\%$ from Eq. (12).

![Fig. 7. Minimum mixer temperature $T_m$ as a function of the concentrated stream surface area $\sigma_c$ divided by $^4$He flow rate $n_1$ for an ideal continuous heat exchanger. The solid curve is for the Kapitza resistivities, $\rho_1$ and $\rho_2$, the same as for copper. The dashed and broken curves are for both $\rho_1$ and $\rho_2$ reduced by the factors 1.8 and 20/3, respectively. The still temperature $T_s$ is 0.7 K and $\sigma_c/\sigma_s=1.6$. At the low temperature end, the curves behave nearly as $T_m \propto (\sigma_c/\sigma_s)^{-1}$. ○—Anderson; □—Wheatley.](image)

In Fig. 7, the minimum mixer temperature, assuming zero heat leak, that can be reached using an ideal continuous heat exchanger, is shown as a function of the quantity $\sigma_c/\eta_3$, where $\sigma_c$ is the total heat transfer surface in contact with the concentrated stream. The ratio $\sigma_c/\sigma_s$ is 1.6, where $\sigma_s$ is the total dilute side heat transfer surface, but there is little sensitivity to this ratio in the range 1–2 because $\rho_c$ is the dominant resistivity. When the ratio is 2, $T_m$ is about 1% less and for a ratio of 1, $T_m$ is about 4% higher; thus, increasing only $\sigma_c$ is of little value in trying to reduce $T_m$. The curve of $T_m$ as a function of $x$ is similar in shape to the curve of $T_m$ as a function of $\sigma_c/\eta_3$.

Mixer temperatures obtained by Anderson\cite{18} and Wheatley \cite{19} using coaxial tube exchangers are shown on Fig. 7. The error bars indicate uncertainty of $\eta_3$. Anderson's result for $T_m$ is higher than the calculated curve. This can be due to a large heat leak, but more likely it is due to an actual temperature difference between the mixer and externally mounted CMN thermometer. Temperatures in Wheatley's refrigerator were measured inside the mixer and are lower than the calculated curve. Such behavior can be explained if $\rho_c$ has a value which is one-half that used in this calculation. The value of $\rho_2$ could also be reduced by a factor of 2 but the results for $T_m$ are rather insensitive to
Before a more detailed comparison can be made between the calculated and experimental performance of the coaxial heat exchangers, measurements are required of the Kapitza resistances of Cu–Ni and stainless steel tubes as well as more experimental data for $T_m$.

In many of the refrigerators presently in operation, the heat exchangers are either coaxial tube or discrete from the still to the mixer. Discrete exchangers are generally more efficient than the coaxial exchanger in terms of heat transferred per volume of liquid in the exchanger, even though they are less efficient in terms of heat transferred per unit surface area. Thus the discrete exchanger, with its high surface area per unit liquid volume, should always be used whenever practically possible. The calculations show, however, that only a 10 cm length of coaxial exchanger made of the 1 and 3 mm tubing discussed above is sufficient to cool the incoming He from 0.7 to 0.2 K at a flow rate of $2 \times 10^{-5}$ moles/sec. This length of coaxial exchanger has a $\sigma$ of 2.4 cm². A sintered copper powder heat exchanger with a surface area even as large as 6 cm² would only be a cube 2 mm on a side. The tubes connecting this tiny exchanger to the still and next exchanger can easily have the same volume that a coaxial exchanger would have. Hence, at least from $T_s=0.7$ to 0.2 K it is much more practical to use a simple coaxial exchanger.

The uncertainty in $\rho$, as small as it may be, is not known with complete accuracy. The conductances and the exchangers generally in use are important since the results depend on the temperature of the calculations.

There is only one measure of the Kapitza $\rho$ resistance, and that of copper–He resistivity of copper–He resistivity. The lower for oxygen and $\rho$ resistivity is given by a lower for oxidized copper–He resistivity in the literature. However, at pressures of $10^4$ atm, the Kapitza resistance might vary from $10^2$ to $10^4$.

More calculations are needed to determine optimum sizes for heat exchanger geometries and temperatures.

The authors are grateful to J. W. Marlow for helpful discussion.

The analytic form of the specific heat, $C_m$, is

$$C_m = 107.167 T + 1.100745 T^2 - 1.063902 T^3$$

and

$$C_m = -6.25023 T + 1.100745 T^2 - 1.063902 T^3$$


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<th>$H_a$ (J/moles)</th>
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use a simple coaxial tube exchanger such as Ehnholm et al.\textsuperscript{4} have done.

The uncertainties of the specific heats of the liquids are as small as a few percent. The conductivities of the liquids are not known nearly so accurately, but in most cases the exact values of the conductivities are not critical. The conductances are so large in the low temperature discrete exchangers generally used that the exact value is unimportant since the liquid in the exchanger is nearly uniform in temperature. The dominating uncertainties in the calculations are the magnitudes of the Kapitza resistivities. There is only one measurement of the copper-dilute solution resistivity.\textsuperscript{5} There are two measurements of the copper-\textsuperscript{3}He resistivity\textsuperscript{12,22} which essentially agree. Anderson\textsuperscript{23} reports that the Kapitza resistivity is a factor of 4 lower for oxidized copper surfaces than it is for surfaces heated in a hydrogen atmosphere. This report, plus the wide variation of \textsuperscript{4}He boundary resistivities\textsuperscript{20} reported in the literature, indicate that \(\rho_1\) and \(\rho_0\) could vary widely.

However, at present it is not known to what extent the Kapitza resistivities of actual heat exchanger surfaces might vary from those used in these calculations.

More calculations are in progress to determine the optimum sizes for discrete exchangers operating at low temperatures.

ACKNOWLEDGMENT

The authors wish to thank David E. Dancy for many helpful discussions.

APPENDIX

The analytic expressions used in this work for the \textsuperscript{4}He specific heat, \(C_v\), in the dilute side of the heat exchangers are

\[
C_v = 107.167 + 6.1 \times 10^7 T^4 - 3.595619965 \times 10^5 T^5 - 1.00745404 \times 10^{97} + 1.755839768 \times 10^{99} T^6 - 2.73191224 \times 10^{10} T^7 \text{ J/mole-K, } (T \leq 0.12 \text{ K};
\]

and

\[
C_v = -6.25052327 + 500.62000727 - 1.4675777843 \times 10^{17} T^2 + 3.842834281 \times 10^{19} T^3 - 5.571748399 \times 10^{21} T^4 + 4.208126854 \times 10^{23} T^5 - 1.281475180 \times 10^{25} T^6 \text{ J/mole-K, } (0.12 < T \leq 0.5 \text{ K}).
\]

The above expressions are valid for a mixer temperature of 0 K, but for a mixer temperature of 20 mK the maximum error is still less than 2\textsuperscript{6}. The enthalpy of the dilute stream for \(T_m = 0\) is given by

\[
H_d = \int_0^T C_d dT.
\]

The enthalpy \(H_d\) of the concentrated stream is just that of pure \textsuperscript{4}He as tabulated previously by Radebaugh.\textsuperscript{15} Values of \(H_d, H_d/T, C_d,\) and \(H_d/T^3\) for several temperatures should be useful in heat exchanger calculations and are listed in Table III. All the enthalpies are for 1 mole of \textsuperscript{4}He in solution. For \(T_m > 0\), the integration in Eq. (A1) is from \(T_m\) to \(T\), and a constant equal to the enthalpy on the solubility curve\textsuperscript{15} must be added.

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20. J. C. Wheatley, private communication.
Journal Title: Review of Scientific Instruments
Volume: 42
Issue:
Month/Year: 1971
Pages: 1111-1119

Article Author: Siegwarth, J.D. et al
Article Title: Analysis of Heat Exchangers for Dilution Refrigerators

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